LU-Decomposition of a Matrix with Entries of Different Kinds

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ABSTRACT

Let $\mathbf{F} \supset \mathbf{K}$ be fields, and consider a matrix A over $\mathbf{F}$ whose entries not belonging to $\mathbf{K}$ are algebraically independent transcendentals over $\mathbf{K}$. It is shown that if $\det A \in \mathbf{K}^*$ $(=\mathbf{K}-\{0\})$, the matrix A, with suitable permutations of its rows and columns, is decomposed into LU-factors with the entries of the U-factor belonging to $\mathbf{K}$.

1. INTRODUCTION

Let $\mathbf{K}$ be a field and $\mathbf{F}$ $(\supset \mathbf{K})$ an extension field. For $S \subset \mathbf{F}$ we denote by $M(S)$ the set of matrices with entries belonging to S. Suppose an n-by-n matrix $A=(A_{ij}) \in M(\mathbf{F})$ is expressed as

$$A = Q + T, \tag{1}$$

where

(i) $Q \in M(\mathbf{K})$,
(ii) nonzero entries of T are algebraically independent transcendentals over $\mathbf{K}$.[1]

In the following we shall denote by T^* the set of nonzero entries of T.

As is well known, A is invertible in the ring $\mathbf{K}[T^*]$ of polynomials in T^* over $\mathbf{K}$—i.e., $A^{-1} \in M(\mathbf{K}[T^*])$—iff $\det A \in \mathbf{K}^*$ $(=\mathbf{K}-\{0\})$. Here we are interested in whether we can compute A^{-1} by means of pivot operations in $\mathbf{K}[T^*]$; moreover, how simple we can make the LU-factors of A by applying suitable permutations to its rows and columns.

[1] That is, none of the nonzero entries of T is algebraically dependent over K on the others.

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By way of illustration, we start with an example. Let $\mathbf{K} = \mathbf{Q}$ (the field of rational numbers), and set $\mathbf{F} = \mathbf{Q}(x, y, z)$ where $\{x, y, z\}$, as a collection, is assumed to be algebraically independent over $\mathbf{Q}$. The matrix

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & x & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & z & 0 \end{bmatrix} \end{array}$$

is expressed as the sum of the following Q and T according to (1):

$$Q = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 \end{bmatrix}.$$

Note that $\det A = 2$ and hence A is invertible in $\mathbf{Q}[x, y, z]$. The matrix A is decomposed into LU-factors in $\mathbf{F}$ as

$$A = LU,$$

with

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -y & y-1 & y-1-\dfrac{2}{x} & 1 & 0 \\ -1 & 2 & 2+\dfrac{1}{x} & -\dfrac{xz+1}{2} & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & x+1 & 1 & 1 \\ 0 & 0 & -x & -1 & 0 \\ 0 & 0 & 0 & -2/x & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

It is observed that some of the entries of L and U, in particular some of the diagonals of U, do not belong to $\mathbf{K}[T^*]$.

However, after rearranging the rows and the columns of A as

$$\tilde{A} = \begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 5 & 3 & 4 & 2 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ y & -1 & 1 & 0 & -1 \\ 1 & 0 & x & 1 & 0 \\ 1 & 0 & 0 & z & 1 \end{bmatrix} \end{array}$$

we obtain the LU decomposition

$$\tilde{A} = \tilde{L}\tilde{U}$$

with

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -y & y-1 & 1 & 0 & 0 \\ -1 & 1 & x/2 & 1 & 0 \\ -1 & 1 & 0 & z & 1 \end{bmatrix}, \qquad \tilde{U} = \begin{bmatrix} -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The LU-factors are much simpler in the sense that all the entries of $\tilde{U}$ are numbers in $\mathbf{K} = \mathbf{Q}$, i.e., $\tilde{U} \in M(\mathbf{K})$, and consequently the entries of $\tilde{L}$ are polynomials in x, y, and z over $\mathbf{K}$ of degree at most 1.

In this paper, we establish a theorem to the effect that this is always the case for any matrix A which admits the expression of (1) with $\det A \in \mathbf{K}^*$, i.e., that it is always possible to find a permutation of rows and one of columns through which the matrix A can be brought to a form decomposable into LU-factors with the U-factor in $M(\mathbf{K})$. Furthermore, it is shown how to find suitable permutations. Some implications of the theorem are also discussed.

2. THE THEOREM

In this section we prove the following theorem.

THEOREM. *Let A be a matrix of the form* (1). *If* $\det A \in \mathbf{K}^*$, *then there exist permutation matrices P_r, P_c and LU-factors $\tilde{L} = (\tilde{L}_{ij})$, $\tilde{U} = (\tilde{U}_{ij})$:*

$$P_r' A P_c = \tilde{L}\tilde{U},$$

such that

(i) $\tilde{L}_{ij}$ *is a polynomial of degree at most* 1 *in nonzero entries* T^* *of* T *over* $\mathbf{K}$ ($\tilde{L}_{ii} = 1$; $\tilde{L}_{ij} = 0$ *for* $i < j$), *and*
(ii) $\tilde{U} \in M(\mathbf{K})$; $\tilde{U}_{ii} \in \mathbf{K}^*$ ($\tilde{U}_{ij} = 0$ *for* $i > j$).

To prove the theorem, the following lemma is crucial, giving a necessary and sufficient condition for a matrix of the form (1) to be invertible in $\mathbf{K}[T^*]$. We shall say that a matrix is *strictly lower triangular* if it is a lower triangular matrix with zero diagonals.

LEMMA 1. *Let* A *be a matrix as in* (1). *Then* $\det A \in \mathbf{K}^*$ *if and only if* $\det Q \neq 0$ *and* $P_r'(TQ^{-1})P_r$ *is strictly lower triangular for some permutation matrix* P_r.

Proof. "*If*" *part*: Suppose $P_r'(TQ^{-1})P_r$ is strictly lower triangular for some permutation matrix P_r. Then, since $\det Q \neq 0$ and $A = Q + T$, we have

$$\begin{aligned}\det A &= \det\left[(I + TQ^{-1})Q\right]\\ &= \det\left[I + P_r'(TQ^{-1})P_r\right]\det Q\\ &= \det Q \in \mathbf{K}^*.\end{aligned}$$

"*Only if*" *part*: If $\det A \in \mathbf{K}^*$, then $\det Q = \det A \neq 0$, so that we may put $S = Q^{-1}$. Suppose, to the contrary, that $P_r'(TS)P_r$ is not strictly lower triangular for any permutation matrix P_r. Then TS has a cycle of nonzero entries, that is, there exist an integer $M \geqq 1$ and a sequence of indices $i(m)$ and $j(m)$ ($m = 1, \dots, M$) such that

$$T_{i(m-1), j(m)} \neq 0 \text{ and } S_{j(m), i(m)} \neq 0 \qquad \text{for} \quad m = 1, \dots, M,$$

where $i(0) = i(M)$. Choose M to be the minimum of such integers. For notational simplicity, we write $T_{i(m-1), j(m)} = t_m$ and $S_{j(m), i(m)} = s_m$.

For $k = 0, 1, \dots,$ consider the expression of the $(j(1), i(1))$ entry of $S(TS)^{kM}$ in the form of the sum of products of T_{ij}'s and S_{ji}'s. Corresponding to the above cycle, it contains a term

$$s_1(s_1 s_2 \cdots s_M)^k (t_1 \cdots t_M)^k,$$

since no other similar terms of $(t_1 \cdots t_M)^k$ exist, due to the minimality of M, and since it cannot be canceled out by nonsimilar terms, by virtue of the algebraic independence of elements of T^*.

Next we formally expand A^{-1} as

$$A^{-1} = \left[(I + TQ^{-1})Q\right]^{-1}$$
$$= S - STS + STSTS - \cdots .$$

Each entry of A^{-1} on the left-hand side is a polynomial in T^* over $\mathbf{K}$ since $\det A \in \mathbf{K}^*$. On the right-hand side, we first observe that each entry of the mth term is a homogeneous polynomial in T^* of degree $m - 1$. Hence, by the algebraic independence of T^*, no cancellation occurs among distinct terms in this expansion.

It follows in particular that the $(j(1), i(1))$ entry of the right-hand side contains a term of arbitrarily high degree, since the nonzero term $(t_1 \cdots t_M)^k$ of degree kM, stemming from $S(TS)^{kM}$ as above, cannot be canceled out for $k = 0, 1, \ldots$. This is a contradiction. ∎

We make use of the following well-known lemma, the proof of which is omitted.

Lemma 2. *If* $\det Q \neq 0$, *then for any permutation matrix* P_r, *there exists a permutation matrix* P_c *and LU-factors* M, $\tilde{U}$ *such that*

$$P_r' Q P_c = M\tilde{U},$$

where M *is a lower triangular matrix with unit diagonals in* $M(\mathbf{K})$, *and* $\tilde{U}$ *a nonsingular upper triangular matrix in* $M(\mathbf{K})$.

With Lemmas 1 and 2, the Theorem is easy to establish as shown below.

Proof of Theorem. Let P_r and P_c be permutation matrices as in Lemmas 1 and 2, respectively. Then from Lemma 2 we obtain

$$\begin{aligned}\tilde{A} &= P_r' A P_c \\ &= P_r'(Q + T)P_c \\ &= \left[I + P_r'(TQ^{-1})P_r\right](P_r' Q P_c) \\ &= \left[I + P_r'(TQ^{-1})P_r\right]M\tilde{U} \\ &= \tilde{L}\tilde{U},\end{aligned}$$

where

$$\tilde{L} = \left[I + P_r'(TQ^{-1})P_r\right]M.$$

Since both factors of $\tilde{L}$ are lower triangular matrices with unit diagonals, $\tilde{L}$ is also a lower triangular matrix with unit diagonals, and therefore $\tilde{A} = \tilde{L}\tilde{U}$ is actually the LU-decomposition of $\tilde{A}$. Obviously $\tilde{U}$ belongs to $M(\mathbf{K})$, and consequently the entries of $\tilde{L} = \tilde{A}\tilde{U}^{-1}$ are polynomials in T^* of degree at most 1. ∎

REMARK 1. In parallel with the Theorem, it is likewise possible to find permutations through which A can be brought to a form decomposable into LU-factors in such a way that the L-factor, instead of the U-factor, belongs to $M(\mathbf{K})$.

REMARK 2. Consider a matrix A in $M(\mathbf{F})$. Then it can be written as

$$A = Q + T,$$

where $Q \in M(\mathbf{K})$ and $T \in M(\mathbf{F} \setminus \mathbf{K})$. In general, the nonzero entries T^* of T are not algebraically independent over $\mathbf{K}$, and an LU-decomposition of the abovementioned kind does not necessarily exist even if $\det A \in \mathbf{K}^*$, as is the case with

$$A = \begin{bmatrix} x & 1+x \\ 1-x & -x \end{bmatrix},$$

where $\mathbf{K} = \mathbf{Q}$ and $\mathbf{F} = \mathbf{Q}(x)$.

However, it may happen that the matrix $A_0 = Q + T_0$, where T_0 is obtained from T by replacing its nonzero entries with algebraically independent transcendentals, satisfies the condition $\det A_0 \in \mathbf{K}^*$. Then the Theorem can be applied to A_0, which in turn implies that A itself can be decomposed, with suitable permutations, into LU-factors with the U-factor belonging to $M(\mathbf{K})$.

3. DISCUSSION

When given a matrix A of the form (1) satisfying the condition $\det A \in \mathbf{K}^*$, we can find the suitable permutations P_r and P_c on the basis of Lemmas 1 and

2. P_r can be determined by the zero-nonzero pattern of TQ^{-1}, and P_c by pivoting operations on the matrix Q. Thus both permutations can be found with $O(n^3)$ arithmetic operations in $\mathbf{K}$.

Lemma 1 gives an efficient way, with $O(n^3)$ arithmetic operations in $\mathbf{K}$, for testing whether a matrix A of the form (1) satisfies the condition $\det A \in \mathbf{K}^*$.

The problem dealt with in the present paper arose when the author was investigating the following problem of large-scale system analysis. Let R and C be the set of row and column numbers, respectively, and let $A(I, J)$ denote the submatrix of A corresponding to $I\ (\subset R)$ and $J\ (\subset C)$. For a matrix A of the form (1), it is known [1] (cf. also the concept of 2-block rank in [2]) that we can find, by an efficient algorithm, two subsets $I \subset R$ and $J \subset C$ such that

$$\operatorname{rank} A = \operatorname{rank} A(I, J) + \operatorname{rank} A(R \setminus I, C \setminus J),$$

$$\operatorname{rank} A(I, J) = \operatorname{rank} Q(I, J),$$

and

$$\operatorname{rank} A(R \setminus I, C \setminus J) = \operatorname{rank} T(R \setminus I, C \setminus J),$$

where the rank is considered over $\mathbf{F}$. If we take I and J to be the minimal of such subsets, we have $|I| = |J|$ and

$$\det A(I, J) \in \mathbf{K}^*.$$

The submatrix $A(I, J)$ above meets the condition of the Theorem. This implies that a matrix A of the form (1) with $\det A \neq 0$ can be decomposed, after suitable permutations P_r and P_c, into LU-factors as

$$P_r' A P_c = \tilde{L}\tilde{U},$$

with a lower triangular matrix

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix}$$

with unit diagonals, and a nonsingular upper triangular matrix

$$\tilde{U} = \begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ 0 & \tilde{U}_{22} \end{bmatrix},$$

such that

(i) the entries of $\tilde{L}_{11}$ and $\tilde{L}_{21}$ are polynomials in T^* over $\mathbf{K}$ of degree at most 1;

(ii) $\tilde{U}_{11} \in M(\mathbf{K})$, and the diagonal entries of $\tilde{U}_{22}$ are algebraically independent over $\mathbf{K}$.

This procedure is applied to the iterative solution of a system of linear-nonlinear equations $f(x) = 0$ in real unknown variables x, as follows. Let us suppose that a sequence of approximate solutions are computed by means of the Newton method, which would involve the solution of $J(x)\Delta x = f(x)$ for Δx through the LU-decomposition of $J(x)$, where $J(x)$ is the Jacobian matrix.

Since the nonconstant derivatives of $f(x)$ may vary in value at each iteration, we regard them as being algebraically independent, or in other words, denoting the nonconstant part of $J(x)$ by $T(x)$, we express $J(x)$ in the form (1):

$$J(x) = Q + T(x)$$

with $\mathbf{K} = \mathbf{Q}$ or $\mathbf{K} = \mathbf{R}$ (the field of real numbers). Furthermore we assume that $\det J(x) \in \mathbf{Q}^*$ or $\mathbf{R}^*$.

As the Theorem guarantees, we can obtain the LU-decomposition of $J(x)$:

$$J(x) = L(x)U$$

with

$$\begin{aligned} L(x) &= \left[I + T(x)Q^{-1}\right]M \\ &= M + T(x)U^{-1}, \end{aligned}$$

where $Q = MU$, as above, and the permutation matrices are suppressed for simplicity. Since M and U do not depend on x, they can be computed before the iteration process starts. At each iteration step, only the L-factor $L(x)$ of $J(x)$ is to be computed. Note that U^{-1} on the right-hand side of $L(x)$ does not cost much, since U is triangular. As pointed out in Remark 1 in the previous section, we may alternatively adopt the LU-decomposition $J(x) = LU(x)$ with the L-factor being independent of x.

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